QUADRATIC GRÖBNER BASES FOR SMOOTH 3×3 TRANSPORTATION POLYTOPES

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ABSTRACT. The toric ideals of 3×3 transportation polytopes $T_{\mathbf{rc}}$ are quadratically generated. The only exception is the Birkhoff polytope B_3.

If $T_{\mathbf{rc}}$ is not a multiple of B_3, these ideals even have square-free quadratic initial ideals. This class contains all smooth 3×3 transportation polytopes.

1. INTRODUCTION

1.1. **Motivation**[1]. A lattice polytope $P \subset \mathbb{R}^d$ defines an ample line bundle L_P on a projective toric variety X_P. (See, e.g., [Ful93, §3.4].) If X_P is smooth (the normal fan of P is unimodular), then L_P is very ample, and provides an embedding $X_P \hookrightarrow \mathbb{P}^{r-1}$, where $r = \#(P \cap \mathbb{Z}^d)$. So we can think of X_P as canonically sitting in projective space. The following question [Stu97, Conjecture 2.9] about the defining equations of $X_P \subset \mathbb{P}^{r-1}$ has been around for quite a while, but its origins are hard to track (cf. [BCF$^+$05]).

Question. *Let P be a lattice polytope whose corresponding projective toric variety is smooth. Is the defining ideal I_P generated by quadratics?*

There are two variations of this question (which are of strictly increasing strength).

- Is the homogeneous coordinate ring $\Bbbk[x_1, \ldots, x_r]/I_P$ Koszul?
- Does I_P have a quadratic Gröbner basis?

The last version has a combinatorial interpretation. It asks for the existence of very special, "quadratic" triangulations of P, see §1.3 below.

1.2. **Results.** Simple transportation polytopes provide a large family of smooth polytopes. Yet, the 3×3 Birkhoff polytope B_3 is a non-simple transportation polytope whose ideal is not generated by quadratic polynomials. In this note, we show that in the 3×3 case, this is the only example. In Section 2, we show that B_3 is the only (3×3)-transportation polytope whose ideal is not quadratically generated.

Proposition 1.1. *If $T_{\mathbf{rc}} \neq B_3$, then $I_{T_{\mathbf{rc}}}$ is quadratically generated.*

[1]This motivation is quoted verbatim from [BCF$^+$05].

If P is a 3×3 transportation polytope which is not a multiple of B_3, we can show in Section 3 that these ideals even have quadratic Gröbner bases. This class contains all smooth 3×3 transportation polytopes.

Theorem 1.2. *If $T_{\mathbf{rc}}$ is not a multiple of B_3, then $I_{T_{\mathbf{rc}}}$ has a squarefree quadratic initial ideal.*

Using different methods, Lindsay Piechnik and the first author showed that (among other polytopes) even multiples of B_3 have quadratic triangulations. We believe that odd multiples ≥ 3 allow quadratic triangulations as well.

1.3. Background.

Transportation Polytopes. Let two vectors $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}_{>0}^n$ and $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}_{>0}^m$ with $\sum_{i=1}^n c_i = \sum_{i=1}^m r_i =: s$ be given. The corresponding $(m \times n)$-transportation polytope $T_{\mathbf{rc}}$ is the set of all non-negative $(m \times n)$-matrices $A = (a_{ij})_{ij}$ satisfying

$$\sum_{i=1}^m a_{ik} = c_k \qquad \text{and} \qquad \sum_{j=1}^n a_{lj} = r_l$$

for $1 \leq k \leq n$, $1 \leq l \leq m$. This is a bounded convex polytope with integral vertices (a lattice polytope for short) in $\mathbb{R}^{mn}$. We number the coordinates of $\mathbb{R}^{mn}$ by a_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$. The upper $((m-1) \times (n-1))$-minor of a matrix A in the polytope determines all other entries. Hence, the dimension of $T_{\mathbf{rc}}$ is at most $(m-1)(n-1)$. On the other hand, $a_{ij} = r_i c_j / s$ determines an interior point, so that the dimension is exactly $(m-1)(n-1)$. In what follows, we focus on the case $m = n = 3$.

Toric Ideals. Let $P \subset \mathbb{R}^d$ be a lattice polytope. The point configuration $\mathcal{A} = P \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ defines a ring homomorphism

$$\begin{aligned} \Bbbk[x_1, \ldots, x_r] &\longrightarrow & \Bbbk[t_0^{\pm 1}, \ldots t_d^{\pm 1}] \\ x_i &\longmapsto & t_0 \, \mathbf{t}^{\mathbf{a}_i} := t_0 \, t_1^{a_{1i}} \cdot \ldots \cdot t_d^{a_{di}}. \end{aligned}$$

Its kernel is the homogenous ideal

$$I_P = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \; : \; \sum u_i \mathbf{a}_i = \sum v_i \mathbf{a}_i \, , \; \sum u_i = \sum v_i \rangle.$$

This ideal is called the toric ideal associated to P (see [Stu96, §4]).

The Birkhoff Polytope. The simplest (3×3)-transportation polytope is the Birkhoff polytope B_3 of doubly stochastic matrices, given by $\mathbf{r} = \mathbf{c} = (1, 1, 1)$. The lattice points in B_3 are the six permutation matrices A_σ for $\sigma \in S_3$. If we denote the corresponding variables by x_σ, the toric ideal I_{B_3} is the principal ideal $\langle x_{123} x_{231} x_{312} - x_{132} x_{213} x_{321} \rangle$. So I_{B_3} is not quadratically generated. I_{B_3} has two initial ideals, $\langle x_{123} x_{231} x_{312} \rangle$, and $\langle x_{132} x_{213} x_{321} \rangle$. Geometrically, this correponds to the fact that $B_3 \cap \mathbb{Z}^9$ is a circuit, i.e., a minimal affinely dependent set. B_3 is the convex

hull of the triangle of even permutation matrices together with the triangle of odd permutation matrices. The two triangles meet in their barycenters.

$$(*) \qquad \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix} + \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} + \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

This (up to scaling) unique affine relation yields the equation generating I_{B_3}.

Smooth Polytopes. For a lattice polytope P, the set of zeros in $\mathbb{P}^{r-1}$ of I_P is the toric variety X_P. This variety is smooth if and only if the edge directions at every vertex of P form a lattice basis. Equivalently, X_P is smooth if and only if the normal fan of P is unimodular (See [Ful93, §2.1]). In this case we call P a smooth polytope. In particular, smooth polytopes are simple: every vertex belongs to dimension many facets. (So, the Birkhoff polytope is not smooth.)

Lemma 1.3. *For a transportation polytope* $T_{\mathbf{rc}}$, *the following are equivalent.*

    (1) $X_{T_{\mathbf{rc}}}$ is smooth.
    (2) $T_{\mathbf{rc}}$ is smooth.
    (3) $T_{\mathbf{rc}}$ is simple.
    (4) $\sum_{i \in I} r_i \neq \sum_{j \in J} c_j$ for all non-trivial sets of indices $I \subset [m]$, $J \subset [n]$.

We have not found a proof in the literature. For completeness, we include one here. (Compare the discussion for general flow polytopes in [BSdLV04]. Lemma 1.3 says that in our case, topes and chambers agree.)

Proof. $(1) \Leftrightarrow (2)$ by [Ful93, § 2.1]. The implication $(2) \Rightarrow (3)$ is valid for all lattice polytopes. The converse, $(3) \Rightarrow (2)$ follows from the fact that transportation polytopes arise from a totally unimodular matrix [Sch86, §19].

$(4) \Rightarrow (3)$: Suppose that $T_{\mathbf{rc}}$ has a vertex A that belongs to $\geq (m-1)(n-1)+1$ facets. Then A has at least that many zero entries. Thus, the bipartite graph given by the non-zero entries has $n + m$ vertices and $\leq n + m - 2$ edges. So this graph cannot be connected. Take for I and J the color classes of one component of this graph.

For $(3) \Rightarrow (4)$ we need some preliminary observations. We use the criterion that an inequality $a_{ij} \geq 0$ defines a facet of $T_{\mathbf{rc}}$ if and only if there is an $A \in T_{\mathbf{rc}}$ such that $a_{ij} = 0$ and with all other entries positive.

Now, suppose we are given $I \subset [m]$ and $J \subset [n]$ with $\sum_{i \in I} r_i = \sum_{j \in J} c_j$. Build a matrix $A \in T_{\mathbf{rc}}$ from a vertex A' of the $I \times J$ transportation polytope, and a vertex A'' of the $I^c \times J^c$ transportation polytope. We abbreviate $m' = |I|$, $m'' = |I^c|$, $n' = |J|$, and $n'' = |J^c|$.

$$\begin{array}{c|c|c|} & I & I^c \\ \hline J & A' & 0 \\ \hline J^c & 0 & A'' \\ \hline \end{array}$$

Lemma 1.4. *The inequalities $a_{ij} \geq 0$ for $(i,j) \in I \times J^c \cup I^c \times J$ define facets of $T_{\mathbf{rc}}$.*

Proof. Say, $(i,j) \in I \times J^c$. Start from all positive A' and A''. Add $\varepsilon m'' n'$ to all $I \times J^c$ entries $\neq (i,j)$, and $\varepsilon(m'n'' - 1)$ to all $I^c \times J$ entries. Now modify A' and A'' in order to obtain the old row and column sums. This amounts to finding points in two (non-integral) transportation polytopes. For small enough ε, the resulting matrix will have positive entries away from (i,j). □

Lemma 1.5. *If the inequality $a_{ij} \geq 0$ $((i,j) \in I \times J)$ defines a facet of the $I \times J$ transportation polytope, then it also defines a facet of $T_{\mathbf{rc}}$.*

Proof. Let A' be a matrix whose only zero entry is (i,j), and let A'' be all positive. As before, we can subtract suitable constants from A' and A'', and find all positive matrices to couterbalance row and column sums. □

To wrap it up, if A' and A'' are vertices of their transportation polytopes, the block matrix A belongs to at least $(m-1)(n-1)+1$ facets. Hence, $T_{\mathbf{rc}}$ is not simple. □

Triangulations. In order to show that a toric ideal has a quadratic Gröbner basis, we use the connection to regular triangulations as outlined in [Stu96, §8]. A subset $F \subseteq P \cap \mathbb{Z}^d$ is a face of a triangulation of P if $\mathrm{conv}(F)$ is a simplex of the triangulation; otherwise F is said to be a non-face. Observe that every superset of a non-face is a non-face.

Definition. A regular unimodular triangulation whose minimal non-faces have two elements is called a quadratic triangulation.[2]

The following characterization is a conglomerate of Corollaries 8.4 and 8.9 in [Stu96].

Theorem 1.6. *The defining ideal I_P of the projective toric variety $X_P \subset \mathbb{P}^{r-1}$ has a squarefree initial ideal if and only if P has a regular unimodular triangulation.*

In that case, the corresponding initial ideal is the Stanley-Reisner ideal of the triangulation: $\mathrm{in}(I_P) = \langle x^F \mid F \text{ minimal non-face} \rangle$.

Here, we abbreviate $x^F := \prod_{i \in F} x_i$. In the example of the Birkhoff polytope, there are two (isomorphic) triangulations of B_3. They are

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[2]Simplicial complexes with this non-face property appear in the literature under the names of flag- or clique-complexes.

both regular and unimodular. In one of them, the triangle of even permutation matrices is the minimal non-face, in the other one, the triangle of odd permutation matrices is the minimal non-face.

Using this correspondence, Theorem 1.2 follows from the following theorem which is what we really prove in Section 3.

Theorem 1.7. *If* T_{rc} *is not a multiple of* B_3*, then* T_{rc} *has a quadratic triangulation.*

Paco's Lemma. A tool we use in both proofs are pulling refinements of hyperplane subdivisions. Let $P \subset \mathbb{R}^d$ be a lattice polytope. As before, order the lattice points $P \cap \mathbb{Z}^d = \mathbf{a}_1, \ldots, \mathbf{a}_r$, and the corresponding variables $x_1 \prec \ldots \prec x_r$. Then, the reverse lexicographic term order yields a pulling triangulation of P. These pulling triangulations have a nice recursive structure: the maximal faces are joins of $\mathbf{a}_1$ with faces of the pulling triangulations of those facets of P that do not contain $\mathbf{a}_1$.

We say that a lattice polytope P has facet width 1 if for each of its facets, P lies between the hyperplane spanned by this facet and the next parallel lattice hyperplane.

Proposition 1.8 (Paco's Lemma [San97, OH01, Sul04])**.** *The lattice polytope* P *has facet width* 1 *if and only if every pulling triangulation of* P *is (regular and) unimodular.*

2. Quadratic Generation

The main tools in the proof of Proposition 1.1 are a hyperplane subdivision and matrix addition. We will first exhibit a Gröbner basis which consists of quadratic and cubic binomials. Then we go on to show that the cubic elements can be expressed using quadratic members of the ideal. The resulting quadratic generating set will usually fail to be a Gröbner basis.

A transportation polytope has a canonical regular subdivision into polytopes of facet width 1. We slice T_{rc} along the hyperplanes $a_{ij} = k$. By Proposition 1.8, every pulling refinement of this subdivision will be a regular unimodular triangulation. A non-face F of such a triangulation either contains a pair of matrices which differ by ≥ 2 in one entry (a minimal non-face of cardinality 2), or all of F belongs to the same cell of the hyperplane subdivision.

The ideal $I_{T_{rc}}$ is generated by a Gröbner basis which is parameterized by the minimal non-faces of the given triangulation. (And the degree of a generator equals the cardinality of the corresponding non-face.) So we need to analyze the cells of the hyperplane subdivision. They have the form

$$\mathbf{Z}_{rc}(K) = \{A \in T_{rc} \mid k_{ij} \leq a_{ij} \leq k_{ij} + 1\}$$

for some matrix K with row sums $\mathbf{r}'$ and column sums $\mathbf{c}'$. After translation we get

$$\mathbf{Z_{rc}}(K) - K = \mathbf{Z_{r-r',c-c'}}(0) =: \mathbf{Z_{c-c'}^{r-r'}}.$$

In order to obtain a full-dimensional cell, $\mathbf{r} - \mathbf{r}'$ and $\mathbf{c} - \mathbf{c}'$ must have coefficients 1 or 2. So, up to symmetry, in the (3×3)-case there are only four types of such cells, namely $\mathbf{Z}_{1,1,1}^{1,1,1}$, $\mathbf{Z}_{1,1,2}^{1,1,2}$, $\mathbf{Z}_{1,2,2}^{1,2,2}$, and $\mathbf{Z}_{2,2,2}^{2,2,2}$. In fact, $\mathbf{Z}_{1,1,2}^{1,1,2}$ and $\mathbf{Z}_{1,2,2}^{1,2,2}$ are unimodular simplices, and $\mathbf{Z}_{1,1,1}^{1,1,1} = B_3$ and $\mathbf{Z}_{2,2,2}^{2,2,2}$ are isomorphic as lattice polytopes.

To summarize, $I_{\mathrm{T_{rc}}}$ is generated by quadratic binomials together with cubic binomials that correspond to affine relations à la $(*)$.

Now let us assume $\mathrm{T_{rc}} \neq B_3$, and, say, $\mathrm{Z_{rc}}(K) \cong \mathbf{Z}_{1,1,1}^{1,1,1}$ is a cell in $\mathrm{T_{rc}}$ giving rise to such a cubic equation. Because $\mathrm{T_{rc}} \neq B_3$, at least one of the nine adjacent cells $\mathrm{Z_{rc}}(K - E_{ij})$ has to be in $\mathrm{T_{rc}}$, where E_{ij} is the $(ij)^{\mathrm{th}}$ unit vector. After translation, we are given the relation $(*)$, and we know that (for $i = j = 1$)

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix} \quad \in \quad \mathrm{T_{rc}} - K.$$

But then, we can use the two quadratic relations

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & 1 \\ 1 & & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & \\ 1 & & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

to generate $(*)$. This completes the proof of Proposition 1.1. □

3. Quadratic Gröbner Bases

In the previous section we have seen that all toric ideals associated to transportation polytopes $\mathrm{T_{rc}} \neq B_3$ are quadratically generated. Now we head for the stronger result stated in Theorem 1.2.

We again start by subdividing the polytope into cells by intersection with hyperplanes of the form $a_{ij} = k$, but this time we choose a coarser subdivision to avoid cells isomorphic to the Birkhoff polytope. We show that we can achieve this by taking all hyperplanes $a_{ij} = k$ except for $(i, j) = (1, 1)$ and $(i, j) = (2, 1)$. In a second step we do a pulling triangulation along a vertex order given by a (globally fixed) linear functional. The analysis of the cells was done using the software package `polymake` [GJ05].

3.1. **Hyperplane Subdivision.** Let $\mathrm{T_{rc}}$ be a transportation polytope with row and column sums $\mathbf{r}$ and $\mathbf{c}$, which is not a multiple of B_3. We order the rows and columns in such a way that $r_1 \geq r_2 \geq r_3$ and $c_1 \geq c_2 \geq c_3$. We can assume that $c_1 \geq r_1$, and thus $c_1 > r_3$ because $\mathrm{T_{rc}}$ is not a multiple of B_3.

As before, we subdivide T_{rc} into cells by cutting with the hyperplanes $\{a_{ij} = k_{ij}\}$ except that we omit the $(1,1)$- and the $(2,1)$-entries. Hence, our cells are of the form

$$\overline{Z}_{rc}(K) = \left\{ A \in T_{rc} \;\middle|\; \begin{array}{l} k_{ij} \leq a_{ij} \leq k_{ij} + 1 \\ \text{for } (i,j) = (3,1) \text{ and } 1 \leq i \leq 3,\, 2 \leq j \leq 3 \end{array} \right\}$$

where $K = (k_{ij})_{ij}$ is a (3×3)-matrix with $k_{11}, k_{21} := 0$. Similar to the previous section, we subtract K from T_{rc}, arriving at cells of the form

$$\overline{Z}_{rc}(K) - K = \overline{Z}_{r-r',c-c'}(0).$$

Again, we get $r_3 - r_3', c_2 - c_2', c_3 - c_3' \in \{1,2\}$. Also, we have

$$c_1 - c_1' = c_1 - k_{31} > r_3 - k_{31} \geq r_3 - r_3'.$$

The projection to the a_{11}-a_{21}-plane is described by the inequalities $r_i - r_i' - 2 \leq a_{i1} \leq r_i - r_i'$ for $i = 1, 2$ and $c_1 - c_1' - 1 \leq a_{11} + a_{21} \leq c_1 - c_1'$. Thus, if $r_1, r_2 \geq 2$, there are four cases (cf. Figure 1 on the left). If $r_1 - r_1' = 1$ or $r_2 - r_2' = 1$, there are three and three more cases (cf. Figure 1 on the right). We cannot have $r_1 - r_1' = r_2 - r_2' = 1$ because $c_1 - c_1' > r_3 - r_3'$.
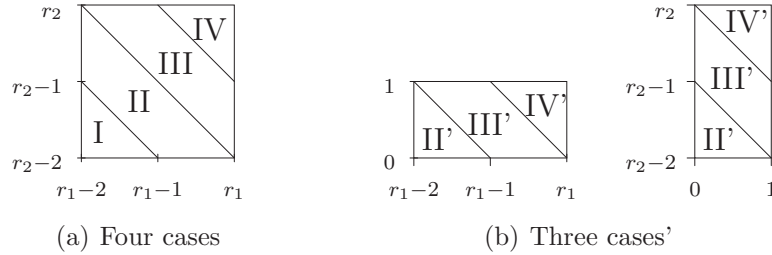

(a) Four cases   (b) Three cases'

Figure 1: The projection onto the a_{11}-a_{21}-plane

If we subtract the lower bounds for a_{11} and a_{21}, we obtain the following 20 translation classes of cells in the subdivision. (We list them in the form $(r - r')\,(c - c')$.)

I $(2,2,1)(1,2,2)$
II $(2,2,1)(2,2,1)$, $(2,2,1)(2,1,2)$, $(2,2,2)(2,2,2)$
III $(2,2,1)(3,1,1)$, $(2,2,2)(3,2,1)$, $(2,2,2)(3,1,2)$
IV $(1,1,2)(2,1,1)$
II' $(2,1,1)(1,2,1)$, $(2,1,1)(1,1,2)$, $(2,1,2)(1,2,2)$,
    $(1,2,1)(1,2,1)$, $(1,2,1)(1,1,2)$, $(1,2,2)(1,2,2)$
III'$(2,1,1)(2,1,1)$, $(2,1,2)(2,2,1)$, $(2,1,2)(2,1,2)$,
    $(1,2,1)(2,1,1)$, $(1,2,2)(2,2,1)$, $(1,2,2)(2,1,2)$
IV' same as IV.

3.2. **Triangulating the Cells.** According to Lemma 1.8, any pulling refinement of our cell decomposition will be unimodular. The subtle part is to devise a pulling order so that the resulting triangulation is flag, i.e., so that the minimal non-faces have cardinality two.

Just as before, a non-face of such a triangulation either contains a non-face of cardinality 2, or it belongs to the same cell of the hyperplane subdivision. Hence, it suffices to guarantee that the induced traingulations of the cells are flag. To achieve this, we order the vertices by decreasing values of the linear functional

$$\mathbf{v} := \begin{bmatrix} 4 & 6 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The induced triangulations of the cells are the pulling triangulations in the induced vertex ordering.

3.2.1. *Description of the Cells.* We give geometric descriptions for all possible cells. For most of them, any triangulation is unimodular and flag. There are two combinatorial types where we have to be careful. We list the vertices of those cells explicitely in Table 1 in the form $[a_{11}\ a_{12}\ a_{21}\ a_{22}]$.

$\overline{\mathbf{Z}}^{1,1,2}_{2,1,1}$, $\overline{\mathbf{Z}}^{1,2,1}_{1,1,2}$, $\overline{\mathbf{Z}}^{1,2,1}_{1,2,1}$, $\overline{\mathbf{Z}}^{1,2,2}_{1,2,2}$, $\overline{\mathbf{Z}}^{2,1,1}_{1,1,2}$, $\overline{\mathbf{Z}}^{2,1,1}_{1,2,1}$, $\overline{\mathbf{Z}}^{2,1,2}_{1,2,2}$, *and* $\overline{\mathbf{Z}}^{2,2,1}_{1,2,2}$. These cells already are unimodular simplices.

$\overline{\mathbf{Z}}^{2,2,1}_{2,1,2}$, $\overline{\mathbf{Z}}^{2,2,1}_{2,2,1}$, $\overline{\mathbf{Z}}^{2,2,2}_{3,1,2}$, *and* $\overline{\mathbf{Z}}^{2,2,2}_{3,2,1}$. These cells are pyramids over a triangular prism $\Delta_2 \times \Delta_1$. All six trianglulations of such a polytope are unimodular and flag.

$\overline{\mathbf{Z}}^{1,2,2}_{2,1,2}$, $\overline{\mathbf{Z}}^{1,2,2}_{2,2,1}$, $\overline{\mathbf{Z}}^{2,1,2}_{2,2,1}$, *and* $\overline{\mathbf{Z}}^{2,1,2}_{2,1,2}$. These cells are a join of an edge and a unit square. Both trianglulations of such a polytope are unimodular and flag.

$\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$ *and* $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$. These cells are isomorphic to a Birkhoff polytope B_3 where we have relaxed one facet. Their vertices are listed in Table 1(a) and 1(b).

$\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$ *and* $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$. These cells are combinatorial duals of the cells in the previous paragraph. Their vertices are in Tables 1(c) and 1(d).

3.2.2. *Triangulating the Interesting Cells.* It remains to show that the pulling triangulations of $\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$, $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$, $\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$, and $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$ given by the order in Table 1 are flag. The vertex facet incidences of these cells are listed in Table 2 which was generated by `polymake` [GJ05]. We will repeatedly use the following fact.

Lemma 3.1. *Suppose that all but two facets F_0 and F_1 of the cell Z contain the vertex v_0. Pull v_0, and choose any triangulation of Z refining this subdivision. Then every minimal non-face with more than two elements belongs to F_0 or to F_1.*   □

Table 1: Birkoff with one relaxed facet and its dual. We record the value of our linear functional in front of each vertex.

(a) $\overline{\mathbf{Z}}_{2,1,1}^{1,2,1}$	(b) $\overline{\mathbf{Z}}_{2,1,1}^{2,1,1}$	(c) $\overline{\mathbf{Z}}_{2,2,2}^{2,2,2}$	(d) $\overline{\mathbf{Z}}_{3,1,1}^{2,2,1}$
$7 : [1\ 0\ 0\ 1]$	$11 : [2\ 0\ 0\ 1]$	$13 : [1\ 1\ 0\ 1]$	$11 : [2\ 0\ 0\ 1]$
$6 : [1\ 0\ 1\ 1]$	$10 : [1\ 1\ 0\ 0]$	$11 : [2\ 0\ 0\ 1]$	$10 : [2\ 0\ 1\ 1]$
$5 : [0\ 1\ 1\ 0]$	$9 : [1\ 1\ 1\ 0]$	$9 : [1\ 1\ 1\ 0]$	$9 : [1\ 1\ 1\ 0]$
$4 : [0\ 1\ 2\ 0]$	$8 : [2\ 0\ 0\ 0]$	$8 : [0\ 1\ 1\ 1]$	$8 : [1\ 1\ 2\ 0]$
$3 : [1\ 0\ 1\ 0]$	$7 : [1\ 0\ 0\ 1]$	$7 : [1\ 0\ 0\ 1]$	$7 : [2\ 0\ 1\ 0]$
$2 : [0\ 0\ 1\ 1]$	$5 : [0\ 1\ 1\ 0]$	$6 : [1\ 0\ 1\ 1]$	$6 : [1\ 0\ 1\ 1]$
$-2 : [0\ 0\ 2\ 0]$	$3 : [1\ 0\ 1\ 0]$	$5 : [0\ 1\ 1\ 0]$	$4 : [0\ 1\ 2\ 0]$
		$4 : [0\ 1\ 2\ 0]$	$2 : [1\ 0\ 2\ 0]$

Table 2: Vertex facet incidences of the interesting cells

(a) $\overline{\mathbf{Z}}_{2,1,1}^{1,2,1}$	(b) $\overline{\mathbf{Z}}_{2,1,1}^{2,1,1}$	(c) $\overline{\mathbf{Z}}_{2,2,2}^{2,2,2}$	(d) $\overline{\mathbf{Z}}_{3,1,1}^{2,2,1}$
$[v_0\ v_1\ v_2\ v_3\ v_4]$	$[v_0\ v_1\ v_2\ v_3]$		
$[v_0\ v_1\ v_2\ v_3\ v_5]$	$[v_0\ v_1\ v_3\ v_4]$	$[v_0\ v_1\ v_3\ v_4\ v_5]$	$[v_0\ v_1\ v_2\ v_3\ v_4]$
$[v_0\ v_1\ v_4\ v_5\ v_6]$	$[v_0\ v_1\ v_2\ v_4\ v_5]$	$[v_0\ v_1\ v_2\ v_4\ v_6]$	$[v_0\ v_1\ v_2\ v_3\ v_5\ v_6]$
$[v_0\ v_2\ v_4\ v_5\ v_6]$	$[v_0\ v_2\ v_3\ v_6]$	$[v_0\ v_3\ v_4\ v_6]$	$[v_0\ v_1\ v_4\ v_5\ v_7]$
$[v_1\ v_3\ v_5\ v_6]$	$[v_0\ v_2\ v_4\ v_5\ v_6]$	$[v_0\ v_1\ v_2\ v_3\ v_5\ v_7]$	$[v_0\ v_2\ v_4\ v_5\ v_6\ v_7]$
$[v_1\ v_3\ v_4\ v_6]$	$[v_0\ v_3\ v_4\ v_6]$	$[v_0\ v_2\ v_3\ v_6\ v_7]$	$[v_1\ v_3\ v_4\ v_7]$
$[v_2\ v_3\ v_4\ v_6]$	$[v_1\ v_2\ v_3\ v_5\ v_6]$	$[v_1\ v_2\ v_4\ v_5\ v_6\ v_7]$	$[v_1\ v_3\ v_5\ v_6\ v_7]$
$[v_2\ v_3\ v_5\ v_6]$	$[v_1\ v_3\ v_4\ v_5\ v_6]$	$[v_3\ v_4\ v_5\ v_6\ v_7]$	$[v_2\ v_3\ v_4\ v_6\ v_7]$

Triangulation of the Cell $\overline{\mathbf{Z}}_{2,1,1}^{1,2,1}$. Looking at the vertex facet incidences in Table 2(a), we see that all facets opposite vertex v_0 are simplices. Hence, after pulling at vertex v_0 we obtain a simplicial complex consisting of the 4 simplices listed in Table 3(a). Pulling at the other vertices does not change the complex anymore, and its only minimal missing faces are the edges (v_1, v_2) and (v_4, v_5).

Triangulation of the Cell $\overline{\mathbf{Z}}_{2,1,1}^{2,1,1}$. The two facets opposite vertex v_0 are pyramids over the square (v_1, v_3, v_5, v_6) with apex v_2 and v_4 respectively. According to Lemma 3.1, any refinement of the pulling of v_0 will be flag.

In our case, we obtain four facets, which are given in Table 3(b). The minimal non-faces are the edges (v_2, v_4) and (v_3, v_5).

Table 3: The vertex facet incidences of the interesting triangulations.

(a) $\overline{\mathbf{Z}}_{2,1,1}^{1,2,1}$	(b) $\overline{\mathbf{Z}}_{2,1,1}^{2,1,1}$	(c) $\overline{\mathbf{Z}}_{3,1,1}^{2,2,1}$	(d) $\overline{\mathbf{Z}}_{2,2,2}^{2,2,2}$
$[v_0\,v_1\,v_3\,v_4\,v_6]$	$[v_0\,v_1\,v_4\,v_5\,v_6]$	$[v_0\,v_1\,v_2\,v_6\,v_7]$	$[v_0\,v_1\,v_3\,v_4\,v_7]$
$[v_0\,v_1\,v_3\,v_5\,v_6]$	$[v_0\,v_1\,v_3\,v_4\,v_6]$	$[v_0\,v_3\,v_4\,v_5\,v_7]$	$[v_0\,v_1\,v_3\,v_6\,v_7]$
$[v_0\,v_2\,v_3\,v_4\,v_6]$	$[v_0\,v_1\,v_2\,v_5\,v_6]$	$[v_0\,v_3\,v_4\,v_6\,v_7]$	$[v_0\,v_1\,v_5\,v_6\,v_7]$
$[v_0\,v_2\,v_3\,v_5\,v_6]$	$[v_0\,v_1\,v_2\,v_3\,v_6]$	$[v_0\,v_1\,v_4\,v_5\,v_7]$	$[v_0\,v_2\,v_3\,v_4\,v_7]$
		$[v_0\,v_1\,v_4\,v_6\,v_7]$	$[v_0\,v_2\,v_3\,v_6\,v_7]$

Triangulation of the Cell $\overline{\mathbf{Z}}_{3,1,1}^{2,2,1}$. The vertex facet incidences of this cell are in Table 2(c). There are again only two facets opposite vertex v_0, which are a square pyramid S and a prism over a triangle P. So again, after Lemma 3.1, we are home.

The facets of our triangulation are listed in Table 3(c). The minimal non-faces are the five edges (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_2, v_5), and (v_5, v_6).

Triangulation of the Cell $\overline{\mathbf{Z}}_{2,2,2}^{2,2,2}$. The vertex facet incidences of this cell are in Table 2(d). This time there are three facets S, P_1 and P_2 opposite the vertex v_0. $S = (v_1, v_3, v_4, v_7)$, is a simplex, $P_1 := (v_1, v_3, v_5, v_6, v_7)$ is a square pyramid with apex v_7 and $P_2 := (v_2, v_3, v_4, v_6, v_7)$ is a square pyramid with apex v_3. Pulling at vertex v_0 gives us a decomposition into three cells which are pyramids over S, P_1 and P_2. The vertex v_1 is contained in the base of P_1, hence pulling at 1 decomposes F_1 into two simplices, while F_2 is not affected. Pulling at v_2 decomposes P_2 into two simplices, and we obtain the simplicial complex given in Table 3(d). Pulling at the remaining vertices does not change this complex anymore. This leaves us with the five minimal non-faces (v_1, v_2), (v_2, v_5), (v_3, v_5), (v_4, v_5) and (v_4, v_6).

Hence, after pulling at all vertices we obtain a flag triangulation, and, we have proven the theorem stated in the introduction:

Theorem 1.7. *If* $\mathrm{T_{rc}}$ *is not a multiple of* B_3, *then* $\mathrm{T_{rc}}$ *has a quadratic triangulation.* $\square$

By the arguments given in the introduction, this immediately implies

Theorem 1.2. *If* $\mathrm{T_{rc}}$ *is not a multiple of* B_3, *then* $I_{\mathrm{T_{rc}}}$ *has a square-free quadratic initial ideal.* $\square$

4. Outlook

In some ways, these results come as a little bit of a disappointment. Seeing that the toric ideal of the Birkhoff polytope is *not* quadratically

generated, we started this project in the hope to find a counterexample to the conjectures among 3×3 transportation polytopes.

We know think it is conceivable to adapt the proof of Proposition 1.1 to all smooth transportation polytopes, or maybe even to general flow polytopes – the natural generalization of transportation polytopes. The same seems substantially harder for the triangulation/Gröbner basis result.

In any case, the techniques can be used to improve known degree bounds be it for sets of generators or for Gröbner bases: it is sufficient to bound the degrees within the cells.

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